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Transcendence of Zeros of Eisenstein Series and Other Modular Functions

by

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1. Introduction

For an even integer $k \geq 4$, let

$$E_k(z) = \frac{1}{2} \sum_{(c,d)} \frac{1}{(cz+d)^k} \quad (z \in \mathcal{H})$$

be the normalized Eisenstein series of weight k with respect to $\Gamma_1 := SL_2(\mathbf{Z})$, where the summation extends over all coprime pairs of integers c and d and $\mathcal{H}$ denotes the complex upper half-plane.

Let

$$\mathcal{F} := \left\{ z = x + iy \in \mathcal{H} \,|\, |z| \geq 1,\ -\frac{1}{2} \leq x \leq 0 \right\} \cup \left\{ z \in \mathcal{H} \,|\, |z| > 1,\ 0 < x < \frac{1}{2} \right\}$$

be the standard fundamental domain for the action of Γ_1 on $\mathcal{H}$.

Elementary considerations show that $E_k(i) = 0$ if $k \equiv 2 \pmod 4$ and $E_k(\rho) = 0$ if $k \not\equiv 0 \pmod 3$ where $\rho := e^{2\pi i/3}$. Otherwise the zeros of E_k on $\mathcal{F}$ are rather mysterious.

In [5] F. K. C. Rankin and H. P. F. Swinnerton-Dyer proved that surprisingly all those zeros lie on the unit circle, i.e. lie on the arc $\{z \in \mathcal{H} \,|\, |z| = 1, -\frac{1}{2} \leq x \leq 0\}$ and are all simple. Their arguments also show that $E_k(i) = 0$ if and only if $k \equiv 2 \pmod 4$ and $E_k(\rho) = 0$ if and only if $k \not\equiv 0 \pmod 3$.

Recently N. Kanou [4] showed that if $k = 12$ or $k \geq 16$, then at least one of the zeros of E_k on $\mathcal{F}$ must be transcendental. His argument is neat and simple and only uses the theory of complex multiplication and the classical result of Schneider on the transcendence of special values of the modular function j, coupled with the fact that $\frac{2k}{B_k}$ (where B_k is the k-th Bernoulli number) for $k = 12$ or $k \geq 16$ is not an integer. (Observe that $-\frac{2k}{B_k}$ is the coefficient at $q = e^{2\pi iz}$ in the q-expansion of E_k.)

In this short note we would like to point out that with a little bit more effort one can in fact prove that *all* the zeros of E_k on $\mathcal{F}$ (with the possible exceptions of i and ρ) are transcendental (sect. 2, Thm. 1). Except for the above mentioned result of Schneider and the theory of complex multiplication, the result of [5] on the location of the zeros of E_k on $\mathcal{F}$ is crucial for our proof.

Note that in the case of Drinfeld modules a corresponding result (using a similar reasoning) has been given by G. Cornelissen [3].

The same argument can also be applied in the case of other modular funtions, we shall give an example below (sect. 2, Thm. 2).

2. Statement of results and proofs

THEOREM 1. *Let z_0 be a zero of E_k. Then if z_0 is not equivalent to i or ρ, z_0 is transcendental.*

Proof. Let

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} \quad (z \in \mathcal{H})$$

be the usual discriminant function of weight 12 and recall that the modular function j is given by

$$j = \frac{E_4^3}{\Delta}.$$

The function

$$g := \frac{E_k^{12}}{\Delta^k}$$

has weight zero, is holomorphic on $\mathcal{H}$ and has a pole of order $12k$ at infinity. Also g has rational Fourier coefficients. Hence

$$g = P(j) \tag{1}$$

where $P(X)$ is a polynomial of degree $12k$ with rational coefficients.

Suppose that $E_k(z_0) = 0$, $z_0 \in \mathcal{F}$. Then by (1), $j(z_0)$ is algebraic. By Schneider's theorem (cf. [6]), if z_0 is not transcendental, then z_0 must be imaginary quadratic, i.e. is the solution of a quadratic equation

$$az^2 + bz + c = 0 \quad (a, b, c \in \mathbf{Z},\ gcd(a, b, c) = 1,\ a > 0,\ b^2 - 4ac = D)$$

where $D < 0$ is a fixed integer.

Recall that the group Γ_1 operates on such points z (the discriminant $D < 0$ being fixed) and that the map

$$z \mapsto \mathbf{Z} \oplus \mathbf{Z}\frac{b + i\sqrt{|D|}}{2a}$$

induces a bijection between their equivalence classes and the equivalence classes of proper $\mathcal{O}_D$-ideals of the imaginary quadratic field $K := \mathbf{Q}(\sqrt{D})$, where $\mathcal{O}_D$ is the order of K of conductor f and we have written $D = D_0 f^2$ with D_0 the fundamental discriminant of K and $f \in \mathbf{N}$.

By the theory of complex multiplication, the values $j(z)$ are algebraic and are all conjugate over K. Put

$$z_1 := \begin{cases} \frac{i\sqrt{|D|}}{2}, & \text{if } D \equiv 0 \pmod 4 \\ \frac{-1+i\sqrt{|D|}}{2}, & \text{if } D \equiv 1 \pmod 4 \end{cases}$$

so that (the class of) z_1 corresponds to the trivial class in the class-group, and let σ be the automorphism over K that sends $j(z_0)$ to $j(z_1)$.

Since $P(j(z_0)) = 0$ and $P(X)$ has rational coefficients, applying σ it follows that $P(j(z_1)) = 0$. From (1) we therefore find that $E_k(z_1) = 0$, and from the result of [5] it then follows that $|z_1| = 1$, $-\frac{1}{2} \le x_1 \le 0$. Hence $D = -4, -3$ and so $z_0 = z_1 = i, \rho$. This concludes the proof of Theorem 1.

One can apply the above reasoning in other similar situations, too. For example, for $n \in \mathbf{N}$ put

$$j_n := (j - 744)|T(n)$$

where $T(n)$ is the usual n-th Hecke operator acting in weight zero.

It is easy to see that j_n is the unique modular function of weight zero with respect to Γ_1 which is holomorphic on $\mathcal{H}$ and whose q-expansion starts

$$j_n(z) = q^{-n} + \mathcal{O}(q)\,.$$

Therefore, by a simple induction argument, j_n is a monic polynomial in j with integral coefficients.

The functions j_n play an important role in [2] and [7].

It was proved in [1] that all the zeros of j_n $(n \ge 1)$ on $\mathcal{F}$ lie on the open arc $\{z \in \mathcal{H} \mid |z| = 1,\ -\frac{1}{2} < x < 0\}$. Hence again in a similar way as above we find

THEOREM 2. *With the above notation, all the zeros of the modular functions j_n $(n \ge 1)$ are transcendental.*

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